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The Hele–Shaw problem with surface tension in a half-plane

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Abstract

In this paper, we consider the Hele–Shaw problem in a 2-dimensional fluid domain $\Omega(t)$ which is constrained to a half-plane. The boundary of $\Omega(t)$ consist of two components: $\Gamma_0(t)$ which lies on the boundary of the half-plane, and $\Gamma(t)$ which lies inside the half-plane. On $\Gamma(t)$ we impose the classical boundary conditions with surface tension, and on $\Gamma_0(t)$ we prescribe the normal derivative of the fluid pressure. At the point where $\Gamma_0(t)$ and $\Gamma(t)$ meet, there is an abrupt change in the boundary condition giving rise to a singularity in the fluid pressure. We prove that the problem has a unique solution with smooth free boundary $\Gamma(t)$ for some small time interval.

Keywords: Laplace equation; Free boundary problems; Hele–Shaw problem; Surface tension; Weighted Hölder spaces

1. Introduction

The classical Hele–Shaw problem models the pressure of fluid squeezed between two parallel plate, a small distance apart. The mathematical problem is to determine the evolution of the 2-dimensional fluid domain $\Omega(t)$ and the fluid pressure $p(y,t)$

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$(y \in \Omega(t))$ such that

$$\Delta_y p = 0 \quad \text{in} \quad \Omega(t), \tag{1.1}$$

$$p = \kappa \quad \text{on} \quad \partial\Omega(t), \tag{1.2}$$

$$V_n = -\mu \frac{\partial p}{\partial n} \quad \text{on} \quad \partial\Omega(t), \tag{1.3}$$

where

$$\Omega(0) \quad \text{is given.} \tag{1.4}$$

Here κ is the mean curvature, n is the outward normal, V_n is the velocity of the free boundary $\partial\Omega(t)$ in the direction n, and μ is a positive parameter. We will use the sign convention that convex hypersurfaces have positive mean curvature. In particular, we have $\kappa = 1$ for unit sphere. The existence and uniqueness of a solution, for a general initial smooth domain $\Omega(0)$, for small time interval was proved by Chen [4], Chen et al. [6], Escher and Simonett [7], Bazaliy [1], Prokert [10], and Bazaliy and Friedman [2]. The methods used by these authors are all different.

In the present work, we consider the situation where $\Omega(t)$ is a 2-dimensional domain restricted to the half-plane $R^2_+ = \{(y_1, y_2) : y_1 > 0\}$. The initial domain $\Omega = \Omega(0)$ is such that

$$\begin{aligned} &\Omega \subset R^2_+,\ \partial\Omega = \Gamma_0 \cup \Gamma,\ \Gamma_0 = \{(0, y_2), 0 < y_2 < b\},\\ &\Gamma \subset \{y_1 > 0\},\ b > 0,\ \Gamma \neq \emptyset, \end{aligned}$$

$$\overline{\Gamma} \text{ is in } C^{k+\alpha} \text{ for some integer } k \geqslant 7 \textit{ and } \alpha \in (0, 1). \tag{1.5}$$

Setting

$$\Gamma_0(t) = \partial\Omega(t) \cap \{y_1 = 0\},\ \Gamma(t) = \partial\Omega(t) \cap \{y_1 > 0\} \tag{1.6}$$

we replace (1.2), (1.3) by

$$p = \kappa \quad \text{on} \quad \Gamma(t), \tag{1.7}$$

$$V_n = -\mu \frac{\partial p}{\partial n} \quad \text{on} \quad \Gamma(t) \tag{1.8}$$

and

$$-\frac{\partial p}{\partial n} = g\left(y_2\right) \quad \text{on} \quad \Gamma_0\left(t\right). \tag{1.9}$$

Here, $g\left(y_2\right)$ is a given function representing the flux of fluid across the boundary $\Gamma_0\left(t\right)$; we shall assume that

$$\begin{gathered} g\left(y_2\right) = 0 \quad \text{if} \quad 0 < y_2 < a_0 \quad \text{or} \quad y_2 > b_0, \quad \text{where} \\ 0 < a_0 < b_0 < b, \quad \text{and} \quad g \in C^{k-3+\alpha}\left(\Gamma_0\right). \end{gathered} \tag{1.10}$$

The corresponding problem with no surface tension, i.e., with $p = 0$ on $\Gamma\left(t\right)$, was studied by King et al. [9]. They constructed an explicit solution in an angular domain and used it to study the motion of the corner point of $\Omega\left(t\right)$.

In this paper, we shall establish local (in time) existence of solutions to problem (1.1), (1.4)–(1.10). Our approach resembles the method used by Bazaliy and Friedman [2] for solving of the Hele–Shaw problem (1.1)–(1.4). That method consists of three steps:

(i) Solving a linear problem in $\Omega \times (0, T)$.
(ii) Reducing the free boundary problem to a nonlinear equation $\psi = S\psi$ where ψ represents the unknowns free boundary and pressure and S is a nonlinear operator, and
(iii) Proving that S is a contraction, so that it has a unique fixed point.

A critical assumption made in this paper is that the curvature of the initial curve Γ in (1.5) is in $E^{5+\alpha}_{5+\alpha}$. This means that the curve is "asymptotically linear" near the corner points (roughly speaking, it forms a contact of order 7 with the tangent lines at the corner points). This assumption is quite crucial from a technical point of view, and because of this assumption the corner points of the free boundary do not move during the time interval for which the solution is established.

In Part 1 of this paper we solve the linear problem in $\Omega \times (0, T)$. To this end we shall combine ideas from [2] with recent estimates derived by Bazaliy and Friedman [3] for the special case when Ω is a sector. We begin with the stationary case, then proceed to the time-dependent case with discretized times $t_1, t_2, \ldots, t_n$ $\left(t_j = j\delta, T = n\delta\right)$, and finally take $\delta \to 0$ to obtain the solution of the model problem in $\Omega \times (0, T)$.

In Part 2 we address the nonlinear problem. Analogously to Bazaliy and Friedman [2] we first reduce it a form $A\psi = \Im\left(\psi\right)$ where $\Im\left(\psi\right)$ is a nonlinear function of ψ and A is the linear operator derived in Part 1, i.e., $A^{-1}\Im$ is the solution of the model problem for data $\Im$. Setting $S = A^{-1}\Im$ we shall then prove that the mapping $\psi \to S\psi$, where $S\psi = A^{-1}\Im\left(\psi\right)$, is a contraction, so that it has a unique fixed point.

We conclude this introduction by noting that whereas in [2] all the estimates are given in terms of Sobolev norms, the estimates in the present paper are given in terms of weighted Hölder norms, where the weights depend on the distance to the corner points. Such estimates should enable us to extend the present results to free boundary problems for systems such as in [5].

Part 1. The linear problem

2. Formulation of the problem

Let Ω be a bounded domain in R^2 with $\partial\Omega = \Gamma_0 \cup \Gamma$ where

$$\Gamma_0 = \{(0, x_2) : 0 < x_2 < b\}, \quad \Gamma \subset \{x_1 > 0\}, \quad b > 0, \quad \Gamma \neq \emptyset$$

and $\overline{\Gamma}$ belongs to $C^{k+\alpha}, k \geqslant 7$.

We set $A = (0, 0)$ and assume that, near A, Γ has the form

$$x_2 = \varphi_1(x_1), \quad \varphi_1'(0) > 0.$$

We denote by ω_1 the angle between Γ and the positive x_2-axis at A, so that $\varphi_1'(0) = \cot\omega_1$. Similarly we set $B = (0, b)$ and assume that, near B, Γ has the form

$$x_2 = \varphi_2(x_1), \quad \varphi_2'(0) < 0.$$

We denote by ω_2 the angle between Γ and the negatively oriented x_2-axis at B, so that $\varphi_2'(0) = -\cot\omega_2$.

Let $\Omega_T = \Omega \times (0, T)$, $\Gamma_{0T} = \Gamma_0 \times (0, T)$, $\Gamma_T = \Gamma \times (0, T)$ for any $T > 0$, and set

$$R(x) = \min\{|x - A|, |x - B|\}.$$

We introduce the Banach space $E_s^{k+\alpha,\beta,\alpha}(\Omega_T)$ of functions $u(x, t)$ with finite norm

$$\|u\|_{E_s^{k+\alpha,\beta,\alpha}(\Omega_T)} = \sum_{|\ell|=0}^{k} \left[\sup_{\Omega_T} R^{|\ell|-s}(x)|D_x^\ell u(x,t)| + \langle D_x^\ell u\rangle^{(\alpha)}_{x,s-|l|,\Omega_T} \right.$$
$$\left. + \langle D_x^\ell u\rangle^{(\beta)}_{t,s-|l|,\Omega_T} + [D_x^\ell u]^{(\alpha,\beta)}_{s-|l|,\Omega_T} \right],$$

where

$$\langle v\rangle^{(\alpha)}_{x,s,\Omega_T} = \sup_{(y,t),(x,t)\,\mathrm{in}\,\Omega_T} R^{\alpha-s}(x,y)\,\frac{|v(y,t)-v(x,t)|}{|y-x|^\alpha},$$

$$\langle v\rangle^{(\beta)}_{t,s,\Omega_T} = \sup_{(x,t),(x,\tau)\,\mathrm{in}\,\Omega_T} R^{-s}(x)\,\frac{|v(x,t)-v(x,\tau)|}{|t-\tau|^\beta}$$

and

$$[v]_{s,\Omega_T}^{(\alpha,\beta)} = \sup_{(y,t),(x,\tau)\text{ in }\Omega_T} R^{\alpha-s}(x,y) \times \frac{|v(y,t)-v(x,t)-v(y,\tau)+v(x,\tau)|}{|y-x|^{\alpha}|t-\tau|^{\beta}},$$

where

$$R(x,y) = \min\{R(x), R(y)\} \quad \text{and} \quad \alpha, \beta \in (0,1)$$

and s is a real number. In a similar way we introduce the spaces $E_s^{k+\alpha,\beta,\alpha}(\Gamma_T)$.

We will use the space $C_s^{\alpha,\beta}(\Gamma_T)$, $\alpha, \beta \in (0,1)$, with norm

$$\|u\|_{C_s^{\alpha,\beta}(\Gamma_T)} = \sup_{\Gamma_T} |u(x,t)| + \langle u\rangle_{x,s,\Gamma_T}^{(\alpha)} + \langle u\rangle_{t,s,\Gamma_T}^{(\beta)}.$$

We shall write $u \in M_s^{4+\alpha}(\Gamma_T)$ if $u \in E_{s+3}^{4+\alpha,\alpha/3,\alpha}(\Gamma_T)$, $u_t \in E_s^{1+\alpha,\alpha/3,\alpha}(\Gamma_T)$ and

$$\|u\|_{M_s^{4+\alpha}(\Gamma_T)} = \|u\|_{E_{s+3}^{4+\alpha,\alpha/3,\alpha}(\Gamma_T)} + \|u_t\|_{E_s^{1+\alpha,\alpha/3,\alpha}(\Gamma_T)}.$$

We shall also write $u \in N_s^{4+\alpha}(\Gamma_T)$ if $u \in M_s^{4+\alpha}(\Gamma_T)$ and

$$D_x^3 u \in C_{s+2}^{\alpha,\frac{1+\alpha}{3}}(\Gamma_T) \cap E_s^{1+\alpha,\alpha/3,\alpha}(\Gamma_T),$$

$$D_x^2 u \in C_{s+1}^{\alpha,\frac{2+\alpha}{3}}(\Gamma_T) \cap E_{s+1}^{2+\alpha,\alpha/3,\alpha}(\Gamma_T).$$

In $N_s^{4+\alpha}(\Gamma_T)$ we introduce norm

$$\|u\|_{N_s^{4+\alpha}(\Gamma_T)} = \|u\|_{M_s^{4+\alpha}(\Gamma_T)} + \left\|D_x^3 u\right\|_{C_{s+2}^{\alpha,\frac{1+\alpha}{3}}(\Gamma_T)} + \left\|D_x^2 u\right\|_{C_{s+1}^{\alpha,\frac{2+\alpha}{3}}(\Gamma_T)}.$$

Let $Q(x,D)$ be a second-order linear, positive definite, self-adjoint elliptic operator in $H_0^1(\Gamma)$ with $C^{6+\alpha}$ coefficients and no zero-order term, so that $Q(x,D)1 = 0$.

Consider the following problem for unknown functions $u(x,t)$ (in Ω_T) and ρ (in Γ_T):

$$\Delta u(x,t) = f_0(x,t) \quad \text{in } \Omega_T,$$
$$\partial_t \rho + \mu \partial_n u = f(x,t) \quad \text{on } \Gamma_T,$$

$$
\begin{aligned}
u - Q\,(x,D)\,\rho &= f_1\,(x,t) \quad \text{on} \quad \Gamma_T,\\
\frac{\partial u}{\partial x_1} &= f_2\,(x,t) \quad \text{on} \quad \Gamma_{0T},\\
\rho\,(x,0) &= 0 \quad \text{on} \quad \Gamma,\\
\rho\,(A,t) &= \rho\,(B,t) = 0;
\end{aligned}
\tag{2.1}
$$

here ∂_n is the derivative in the direction of the outward normal n.

We assume that for some $s \geqslant 1$

$$
\begin{aligned}
&f_0 \in E_{s-1}^{\alpha,\alpha/3,\alpha}\,(\Omega_T)\,, \quad f \in E_s^{1+\alpha,\alpha/3,\alpha}\,(\Gamma_T)\,,\\
&f_1 \in E_{s+1}^{2+\alpha,\alpha/3,\alpha}\,(\Gamma_T)\,, \quad f_2 \in E_s^{1+\alpha,\alpha/3,\alpha}\,(\Gamma_{0T})\,,
\end{aligned}
\tag{2.2}
$$

$$
f_0, \; f, \; f_1, \; \text{and} \; f_2 \; \text{vanish} \; as \quad t = 0
$$

and

$$
f_2\,(0,x_2,t) = 0 \quad \text{if} \quad 0 < x_2 < \varepsilon_0 \quad \text{or} \quad 0 < b - x_2 < \varepsilon_0 \tag{2.3}
$$

for some $\varepsilon_0 > 0$.

For the linear problem we shall require that the angles ω_i satisfy the condition

$$
\min\left\{5 + \frac{\pi}{2\omega_i}, 2 + \frac{\pi}{\omega_i}\right\} > 3 + s \quad (i = 1,2)\,. \tag{2.4}
$$

For the analysis of the nonlinear problem we shall take $s = 1+\alpha$ and also require that

$$
\omega_i < \frac{\pi}{2\,(5+\alpha)} \quad (i = 1,2)\,. \tag{2.5}
$$

Note that if the ω_i satisfy (2.5) and $s = 1+\alpha$, $\alpha \in (0,1)$, then the inequalities in (2.4) are satisfied.

The main result of Part 1 is the following theorem.

Theorem 2.1. *Under preceding assumptions on Q and* (2.2)–(2.4), *if T is sufficiently small then there exists a unique solution (u,ρ) to* (2.1) *with*

$$
u \in E_{s+1}^{2+\alpha,\alpha/3,\alpha}(\Omega_T), \quad \rho \in N_s^{4+\alpha}(\Gamma_T); \tag{2.6}
$$

furthermore,

$$\|u\|_{E_{s+1}^{2+\alpha,\alpha/3,\alpha}(\Omega_T)} + \|\rho\|_{N_s^{4+\alpha}(\Gamma_T)}$$
$$\leqslant C\left\{\|f_0\|_{E_{s-1}^{\alpha,\frac{\alpha}{3},\alpha}(\Omega_T)} + \|f\|_{E_s^{1+\alpha,\frac{\alpha}{3},\alpha}(\Gamma_T)} + \|f_1\|_{E_{s+1}^{2+\alpha,\frac{\alpha}{3},\alpha}(\Gamma_T)} + \|f_2\|_{E_s^{1+\alpha,\frac{\alpha}{3},\alpha}(\Gamma_{0T})}\right\}, \tag{2.7}$$

where C is a constant independent of f_0, f, f_1, f_2.

3. The elliptic problem

Consider the elliptic problem for (v, ρ)

$$\begin{aligned} \Delta v &= f_0(x) \quad \text{in } \Omega, \\ \rho + \mu\partial_n v &= f(x) \quad \text{on } \Gamma, \\ v - Q(x, D)\rho &= f_1(x) \quad \text{on } \Gamma, \\ \frac{\partial v}{\partial x_1} &= f_2(x_2) \quad \text{on } \Gamma_0, \\ \rho(A) &= \rho(B) = 0. \end{aligned} \tag{3.1}$$

We introduce the Banach space $E_s^{k+\alpha}(\Omega)$ of functions $u(x)$ with finite norm

$$\|u\|_{E_s^{k+\alpha}(\Omega)} = \sum_{|l|=0}^{k}\left[\sup_{\Omega} R^{|l|-s}(x)\left|D_x^l u(x)\right| + \left\langle D_x^l u\right\rangle_{x,s-|l|,\Omega}^{(\alpha)}\right],$$

where

$$\langle v\rangle_{x,s,\Omega}^{(\alpha)} = \sup_{\Omega} R^{\alpha-s}(x, y)\frac{|v(y) - v(x)|}{|y - x|^{\alpha}}.$$

In a similar way we introduce the spaces $E_s^{k+\alpha}(\Gamma)$, $E_s^{k+\alpha}(\Gamma_0)$, and $N_s^{4+\alpha}(\Gamma)$ as the subset of $N_s^{4+\alpha}(\Gamma_T)$ functions independent of t. We assume that for some real number $s \geqslant 1$

$$f_0 \in E_{s-1}^{\alpha}(\Omega), \; f \in E_s^{1+\alpha}(\Gamma), \; f_1 \in E_{s+1}^{2+\alpha}(\Gamma), \; f_2 \in E_s^{1+\alpha}(\Gamma_0) \tag{3.2}$$

and

$$f_2\left(0, x_2\right)=0 \quad \text{if} \quad 0<x_2<\varepsilon_0 \quad \text{or} \quad 0<b-x_2<\varepsilon_0 \tag{3.3}$$

for some $\varepsilon_0>0$.

Theorem 3.1. *Under the assumptions on* $Q\left(x, D\right)$ *in Theorem* 2.1 *and the assumptions* (2.4), (3.2), (3.3) *there exists a unique solution of* (3.1) *with*

$$v \in E_{s+1}^{2+\alpha}\left(\Omega\right), \quad \rho \in N_s^{4+\alpha}\left(\Gamma\right); \tag{3.4}$$

furthermore,

$$\begin{aligned} & \|v\|_{E_{s+1}^{2+\alpha}(\Omega)}+\|\rho\|_{N_s^{4+\alpha}(\Gamma)} \\ & \quad \leqslant C\left\{\|f_0\|_{E_{s-1}^{\alpha}(\Omega)}+\|f\|_{E_s^{1+\alpha}(\Gamma)}+\|f_1\|_{E_{s+1}^{2+\alpha}(\Gamma)}+\|f_2\|_{E_s^{1+\alpha}(\Gamma_0)}\right\} \end{aligned} \tag{3.5}$$

were C *is a constant independent of* f_0, f, f_1, f_2.

For the sake of clarity we shall first prove the theorem under the additional assumptions

$$\varphi_i'\left(x_1\right)=const.=\varphi_i'\left(0\right) \quad \text{if} \quad 0<x_1<\varepsilon_1 \quad \left(0<\varepsilon_1<\varepsilon_0\right). \tag{3.6}$$

In the next section we shall show how the condition (3.6) may be dropped.

Proof. Consider the quadratic form

$$J\left(w\right)=\frac{1}{2}\int_\Omega|\nabla w|^2+\int_\Gamma\left(\mu^{-1}Q^{-1}w-h\right)w+\int_\Omega f_0 w+\int_{\Gamma_0} f_2 w,$$

where

$$h=\mu^{-1}\left(f+Q^{-1}f_1\right) \quad \text{on} \quad \Gamma.$$

Observe, from (3.1), that if $\rho \in H_0^1\left(\Gamma\right)$ then $\rho=-Q^{-1}\left(f_1-v\right)$. Let v be such that

$$J\left(v\right)=\min_{w \in H^1(\Omega),} J\left(w\right).$$

As in [2] $Q^{-1}w \in H^{5/2}(\Gamma)$ for $w \in H^{1/2}(\Gamma)$,

$$\int_{\Gamma} \left(Q^{-1}w\right) w \geqslant c \, \|w\|^2_{H_0^1(\Gamma)} \quad (c > 0),$$

a minimizer $v \in H^1(\Omega)$ exists, and v is the unique weak solution of (3.1) with $\rho = -Q^{-1}(f_1 - v)$.

Using elliptic regularity one can show, by a bootstrap argument, that v, ρ have the regularity asserted by (3.4) away from the corner points A, B. To prove the asserted regularity at the corner points, say at A, we extend the solution v, ρ from $|x| < \varepsilon_1/4$ into the infinite sector

$$G = \left\{(x_1, x_2) : 0 < x_1 < \infty, \quad x_2 > \varphi_1'(0)\, x_1\right\}$$

in such a way that the extended functions $\widetilde{v}, \widetilde{\rho}$ vanish if $|x| > \varepsilon_1/2$ and they have the same regularity for $|x| > \varepsilon_1/4$ as in (3.4). Then $(\widetilde{v}, \widetilde{\rho})$ forms a solution of (3.1) in G with $\widetilde{f}_0, \widetilde{f}, \widetilde{f}_1, \widetilde{f}_2$ as in (3.2).

By Theorem 2.1 of Bazaliy and Friedman [3] there exists a solution to (2.1) in the case $\Omega = G$ provided all the f's have compact supports and

$$Q(x, D) = c \frac{\partial^2}{\partial r^2} \quad \text{on} \quad x_2 = \varphi_1'(0)\, x_1, \quad x_1 > 0, \tag{3.7}$$

where c is a negative constant, and this solution satisfies (2.7). By Remark 8.2 of Bazaliy and Friedman [3], this solution also satisfies the estimates, for some $\eta > 0$ and $r \to \infty$,

$$|\nabla u| \leqslant \frac{C}{r^{1+\eta}}, \quad |\sigma_r| \leqslant \frac{C}{r^{1+\eta}}. \tag{3.8}$$

By Theorem 8.1 of Bazaliy and Friedman [3], a solution of (2.1) with the properties (2.6), (3.8) is unique.

The proof of the above results for $\Omega = G$ extends to the stationary problem. In fact, the proof in this case is quite simpler provided Q has the form (3.7). However, in our case $Q(x, D)$ does not have the form (3.7). We therefore proceed differently.

Since

$$Q(x, D) = \frac{\partial}{\partial r}\left((c + O(r)) \frac{\partial}{\partial r}\right) \quad \text{near } r = 0 \quad (c < 0) \tag{3.9}$$

(see (7.10) for our case) we can approximate $Q(x, D)$ by $Q_n(x, D)$ such that

$$Q_n(x, D) = \begin{cases} c\dfrac{\partial^2}{\partial r^2} & \text{if} \quad r < 1/n, \\ Q(x, D) & \text{if} \quad r \geqslant 1/n \end{cases}$$

(n large). Let (v_n, ρ_n) denote the minimizer solution (v, ρ) corresponding to Q_n. Then

$$v_n \to v, \quad \rho_n \to \rho \quad \text{in a weak norm}$$

as $n \to \infty$. Note that (v_n, ρ_n) satisfies (3.4). It also satisfies (3.5) and (3.8), but the constant C may depend on n, as $n \to \infty$. We want, in fact, to show that (3.5) holds with C independent of n.

Define $(\widetilde{v}_n, \widetilde{\rho}_n)$ by truncation, as in the paragraph preceding (3.7), and take $(1/n) < \varepsilon_1/4$. Then

$$\widetilde{v}_n - c\frac{\partial^2}{\partial r^2}\widetilde{\rho}_n = \widetilde{f}_{1n} - \left(c\frac{\partial^2}{\partial r^2} - Q_n\right)\widetilde{\rho}_n,$$

where $\widetilde{f}_{1n}$ is defined as $\widetilde{f}_1$ with respect to $Q = Q_n$. Introduce a cutoff function

$$\psi(x) = \begin{cases} 1 & \text{if} \quad |x| < \varepsilon_2, \\ 0 & \text{if} \quad |x| > 2\varepsilon_2, \end{cases}$$

where ε_2 is a positive constant (independent of n). Let $(v, \rho) \equiv (\widehat{v}_n, \widehat{\rho}_n)$ be the solution in G corresponding to the boundary condition

$$v - c\frac{\partial^2}{\partial r^2}\rho = \widetilde{f}_{1n} - \psi(x)\left(c\frac{\partial^2}{\partial r^2} - Q_n(x, D)\right)\rho.$$

Using (3.9) one can prove by the contraction mapping theorem that such a solution exists and is unique, and that it satisfies (3.5) and (3.8) with C independent of n.

Let

$$v_n^* = \widetilde{v}_n - \widehat{v}_n, \quad \rho_n^* = \widetilde{\rho}_n - \widehat{\rho}_n.$$

Then $(v, \rho) \equiv (v_n^*, \rho_n^*)$ is a solution satisfying the boundary condition

$$v - c\rho_{rr} = -\psi(x)\left(c\frac{\partial^2}{\partial r^2} - Q_n(x, D)\right)\rho_n^* - (1 - \psi(x))\left(c\frac{\partial^2}{\partial r^2} - Q_n(x, D)\right)\widetilde{\rho}_n$$

and, by Remark 8.2 in [3], it is uniquely determined in the class (3.8) and it satisfies (3.5) with C independent of n. Hence,

$$\left\|\widetilde{\rho}_n-\widehat{\rho}_n\right\|_{N_s^{4+\alpha}} \leqslant C\left\|rD^2\left(\widetilde{\rho}_n-\widehat{\rho}_n\right)\right\|_{E_{s+1}^{2+\alpha}}+C\left\|rD^2\widetilde{\rho}_n\right\|_{E_{s+1}^{2+\alpha}}.$$

Since

$$\left\|\widehat{\rho}_n\right\|_{N_s^{4+\alpha}} \leqslant C, \quad C \text{ independent of } n,$$

we get

$$\left\|\widetilde{\rho}_n\right\|_{N_s^{4+\alpha}} \leqslant C\left\|rD^2\widetilde{\rho}_n\right\|_{E_{s+1}^{2+\alpha}}+C$$

and by interpolation we can get rid of the first term on the right-hand side. Hence, $\left\|\widetilde{\rho}_n\right\|_{E_{s+3}^{4+\alpha}} \leqslant C$, and taking $n \to \infty$ we get $\|\widetilde{\rho}\|_{E_{s+3}^{4+\alpha}} \leqslant C < \infty$. From this we easily obtain the asserted regularity of (v, ρ) near the corner points, as well as the estimate (3.5) near the corner points.

The corresponding regularity properties of a solution near smooth internal points of Γ in the elliptic and the time-dependent cases follow from [1,6,7].

4. Removing the restriction (3.6)

We shall perform a mapping of Ω (in the variable x) onto a domain $\widetilde{\Omega}$ (in variable $\widetilde{x}$) for which the condition (3.6) is satisfied. To do that we use an argument due to Vasil'eva [11]. Let $\varsigma(x)$ be a cutoff function with support consisting of two small circles of radius ε with centers at A and B. Let $\widetilde{x}=(\widetilde{x}_1, \widetilde{x}_2)$ be the new coordinates such that in a vicinity of A

$$\widetilde{x}_1=x_1, \widetilde{x}_2=x_2-\varsigma(x)\left(\varphi_1(x_1)-\varphi_1'(0)x_1\right),$$

$\widetilde{x}=(\widetilde{x}_1, \widetilde{x}_2)$ is defined similarly in a vicinity of B by the function $\varphi_2(x_1)$, and $\widetilde{x}=x$ elsewhere. By this transform the domain Ω is mapped into $\widetilde{\Omega}$ for which the condition (3.6) is satisfied. In the new variables the equation $\Delta v=f_0$ becomes

$$\begin{aligned}\Delta_{\widetilde{x}}\widetilde{v} = f_0 &- \frac{\partial^2\widetilde{v}}{\partial\widetilde{x}_1^2}\left[\left(\frac{\partial\widetilde{x}_1}{\partial x_1}\right)^2+\left(\frac{\partial\widetilde{x}_1}{\partial x_2}\right)^2-1\right]\\ &-\frac{\partial^2\widetilde{v}}{\partial\widetilde{x}_2^2}\left[\left(\frac{\partial\widetilde{x}_2}{\partial x_1}\right)^2+\left(\frac{\partial\widetilde{x}_2}{\partial x_2}\right)^2-1\right]\end{aligned}$$

$$-2\frac{\partial^2\widetilde{v}}{\partial\widetilde{x}_1\partial\widetilde{x}_2}\left[\frac{\partial\widetilde{x}_1}{\partial x_1}\frac{\partial\widetilde{x}_2}{\partial x_1}+\frac{\partial\widetilde{x}_1}{\partial x_2}\frac{\partial\widetilde{x}_2}{\partial x_2}\right]$$
$$-\frac{\partial\widetilde{v}}{\partial\widetilde{x}_1}\frac{\partial^2\widetilde{x}_1}{\partial x_1^2}-\frac{\partial\widetilde{v}}{\partial\widetilde{x}_2}\frac{\partial^2\widetilde{x}_2}{\partial x_1^2}-\frac{\partial\widetilde{v}}{\partial\widetilde{x}_1}\frac{\partial^2\widetilde{x}_1}{\partial x_2^2}-\frac{\partial\widetilde{v}}{\partial\widetilde{x}_2}\frac{\partial^2\widetilde{x}_2}{\partial x_2^2}=\widetilde{f}_0\left(\widetilde{x},\widetilde{v}\right). \tag{4.1}$$

We can also write the corresponding boundary conditions for $\widetilde{v}$, $\widetilde{\rho}$ and use Theorem 3.1 in $\widetilde{\Omega}$ to prove the existence of a unique solution $\widetilde{v}$, $\widetilde{\rho}$ by the contraction mapping theorem. Indeed, to apply the contraction mapping theorem we have to estimate, for example,

$$J=\left\|\frac{\partial\widetilde{v}}{\partial x}\frac{\partial^2\widetilde{x}}{\partial x^2}\right\|_{E_{s-1}^{\alpha,\alpha/3,\alpha}\left(\widetilde{\Omega}\right)}, \tag{4.2}$$

keeping in mind that our aim is to obtain an estimate of the form

$$J\leqslant C\left(\varepsilon\right)\left\|\frac{\partial\widetilde{v}}{\partial\widetilde{x}}\right\|_{E_s^{1+\alpha,\alpha/3,\alpha}\left(\widetilde{\Omega}\right)}, \tag{4.3}$$

where $C(\varepsilon)\to 0$ as $\varepsilon\to 0$. As an example we consider one term from the right-hand side of (4.2),

$$\sup_{\widetilde{\Omega}} r^{-s+1}\left|\frac{\partial\widetilde{v}}{\partial\widetilde{x}}\frac{\partial^2\widetilde{x}}{\partial x^2}\right| \leqslant \sup_{\widetilde{\Omega}} r^{-s}\left|\frac{\partial\widetilde{v}}{\partial\widetilde{x}}\right|\cdot\sup_{\widetilde{\Omega}} r\left|\frac{\partial^2\widetilde{x}}{\partial x^2}\right|$$
$$\leqslant \left\|\frac{\partial\widetilde{v}}{\partial\widetilde{x}}\right\|_{E_s^{1+\alpha,\alpha/3,\alpha}\left(\widetilde{\Omega}\right)}\cdot\sup_{\widetilde{\Omega}} r\left|\frac{\partial^2\widetilde{x}}{\partial x^2}\right|. \tag{4.4}$$

The function $\frac{\partial^2\widetilde{x}}{\partial x^2}$ in the ε-neighborhood of the point A consists of the terms of the form $\frac{\partial^2\widetilde{x}_2}{\partial x_1^2}=-\varsigma''(x)\left(\varphi_1(x_1)-\varphi_1'(0)\,x_1\right)-2\varsigma'(x)\left(\varphi_1'(x_1)-\varphi_1'(0)\right)-\varsigma(x)\,\varphi_1''(x_1)$. In this neighborhood $\varphi_1(x_1)-\varphi_1'(0)\,x_1=\varphi_1''(\overline{x}_1)\,x_1^2/2=O\left(\varepsilon^2\right)$, $\varphi_1'(x_1)-\varphi_1'(0)=O(\varepsilon)$, $\left|\varphi_1''(x_1)\right|\leqslant const.$, $\left|\varsigma^{(k)}\right|\leqslant const.\,\varepsilon^{-k}$, so that the factor $\left|\frac{\partial^2\widetilde{x}}{\partial x^2}\right|$ is bounded. A similar argument holds in the ε-neighborhood of the point B, and hence the inequality (4.4) gives

$$\sup_{\widetilde{\Omega}} r^{-s+1}\left|\frac{\partial\widetilde{v}}{\partial\widetilde{x}}\frac{\partial^2\widetilde{x}}{\partial x^2}\right|\leqslant const.\ \varepsilon\left\|\frac{\partial\widetilde{v}}{\partial\widetilde{x}}\right\|_{E_s^{1+\alpha,\alpha/3,\alpha}\left(\widetilde{\Omega}\right)}.$$

Similar estimates can be derived for other terms in $\left\| \frac{\partial \widetilde{v}}{\partial x} \right\|_{E_{s-1}^{1+\alpha,\alpha/3,\alpha}\left(\widetilde{\Omega}\right)}$, leading to the inequality (4.3), and hence to the existence of a unique solution $(\widetilde{v}, \widetilde{\rho})$ satisfying the estimates (3.5). The same estimates then hold also for (v, ρ).

5. Proof of Theorem 2.1

We shall use Theorem 3.1 to prove Theorem 2.1 by the finite-difference approximation method used in [2]. Let

$$t_j = j\delta \quad \text{for } j = 0, 1, 2 \ldots, n, \quad n\delta = T$$

and consider the time-discretized system

$$\begin{aligned}
&\Delta u\left(x, t_j\right) = f_0\left(x, t_j\right) \quad \text{in } \Omega, \\
&\frac{\rho\left(x, t_j\right) - \rho\left(x, t_{j-1}\right)}{\delta} + \mu \partial_n u\left(x, t_j\right) = f\left(x, t_j\right) \quad \text{on } \Gamma, \\
&u\left(x, t_j\right) - Q(x, D)\, \rho\left(x, t_j\right) = f_1\left(x, t_j\right) \quad \text{on } \Gamma, \\
&\frac{\partial u\left(x, t_j\right)}{\partial x_1} = f_2\left(x, t_j\right) \quad \text{on } \Gamma_0, \\
&\rho(x, 0) = 0 \quad \text{on } \Gamma.
\end{aligned} \tag{5.1}$$

Applying Theorem 3.1 successively, we deduce that this system has a unique solution and the following estimates hold:

$$\begin{aligned}
&\left\| u\left(\cdot, t_j\right) \right\|_{E_{s+1}^{2+\alpha}(\Omega)} + \left\| \rho\left(\cdot, t_j\right) \right\|_{N_s^{4+\alpha}(\Gamma)} + \left\| \frac{\rho\left(\cdot, t_j\right) - \rho\left(\cdot, t_{j-1}\right)}{\delta} \right\|_{E_s^{1+\alpha}(\Gamma)} \\
&\leqslant C \left\{ \left\| f_0\left(\cdot, t_j\right) \right\|_{E_{s-1}^{\alpha}(\Omega)} + \left\| f\left(\cdot, t_j\right) \right\|_{E_s^{1+\alpha}(\Gamma)} \right. \\
&\quad \left. + \left\| f_1\left(\cdot, t_j\right) \right\|_{E_{s+1}^{2+\alpha}(\Gamma)} + \left\| f_2\left(\cdot, t_j\right) \right\|_{E_s^{1+\alpha}(\Gamma_0)} \right\}.
\end{aligned} \tag{5.2}$$

As in [2] we now form the functions $u_\delta(x, t)$, $\rho_\delta(x, t)$ such, that for $t_j \leqslant t \leqslant t_{j+1}$

$$\begin{aligned}
u_\delta(\cdot, t) &= u\left(\cdot, t_j\right) \frac{t_{j+1} - t}{\delta} + u\left(\cdot, t_{j+1}\right) \frac{t - t_j}{\delta}, \\
\rho_\delta(\cdot, t) &= \rho\left(\cdot, t_j\right) \frac{t_{j+1} - t}{\delta} + \rho\left(\cdot, t_{j+1}\right) \frac{t - t_j}{\delta}.
\end{aligned}$$

Similarly we define functions $f_{0\delta}(x,t)$, $f_\delta(x,t)$, $f_{1\delta}(x,t)$, $f_{2\delta}(x,t)$ and deduce from (5.2) that

$$\begin{aligned}
&\sup_t \|u_\delta(\cdot,t)\|_{E^{2+\alpha}_{s+1}(\Omega)} + \sup_t \left\|\rho_\delta(\cdot,t)\right\|_{N^{4+\alpha}_s(\Gamma)} + \sup_t \left\|\partial_t\rho_\delta(\cdot,t)\right\|_{E^{1+\alpha}_s(\Gamma)} \\
&\quad \leqslant C\left\{\sup_t \|f_{0\delta}(\cdot,t)\|_{E^{\alpha}_{s-1}(\Omega)} + \sup_t \|f_\delta(\cdot,t)\|_{E^{1+\alpha}_s(\Gamma)}\right. \\
&\qquad \left. + \sup_t \|f_{1\delta}(\cdot,t)\|_{E^{2+\alpha}_{s+1}(\Gamma)} + \sup_t \|f_{2\delta}(\cdot,t)\|_{E^{1+\alpha}_s(\Gamma_0)}\right\}. \qquad (5.3)
\end{aligned}$$

Using the compactness theorem in $E^{k+\alpha}_s$ and taking subsequence

$$u_\delta \to u, \quad \rho_\delta \to \rho,$$

we find that (u,ρ) forms a solution of (2.1) and that the estimate (5.3) holds for (u,ρ). The Hölder estimates in t are obtained by considering the system for

$$u(x,t) - u(x,\tau), \quad \rho(x,t) - \rho(x,\tau).$$

It remains to prove uniqueness. Suppose f_0, f, f_1, and f_2 vanish; we wish to prove that u and ρ vanish. Since $\Delta u = 0$

$$\int_{\Omega_T} |\nabla u|^2 = \int_{\Gamma_T} u\,\frac{\partial u}{\partial n} + \int_{\Gamma_{0T}} u\,\frac{\partial u}{\partial n} = -\frac{1}{\mu}\int_{\Gamma_T} Q(x,D)\,\rho\cdot\frac{\partial\rho}{\partial t}.$$

Next

$$\begin{aligned}
\int_{\Gamma_T} \frac{\partial\rho}{\partial t} Q(x,D)\,\rho &= \int_{\Gamma_T}\left\{\frac{\partial}{\partial t}\left[\rho Q(x,D)\,\rho\right] - \rho Q(x,D)\,\frac{\partial\rho}{\partial t}\right\} \\
&= \int_{\Gamma_T}\left\{\frac{\partial}{\partial t}\left[\rho Q(x,D)\,\rho\right] - \frac{\partial\rho}{\partial t} Q(x,D)\,\rho\right\} \\
&= \int_{\Gamma} \rho Q(x,D)\,\rho\,|_{t=T} - \int_{\Gamma_T} \frac{\partial\rho}{\partial t} Q(x,D)\,\rho,
\end{aligned}$$

where we used the self-adjointness of $Q(x,D)$ and the condition $\rho(x,0)=0$. Hence

$$\int_{\Omega_T} |\nabla u|^2 + \frac{1}{2\mu}\int_{\Gamma} \rho Q(x,D)\,\rho\,|_{t=T} = 0,$$

and since $\rho(\cdot, T) \in H_0^1(\Gamma)$, on account of $\rho(A, T) = \rho(B, T) = 0$ the second integral on the left-hand side is >0 if $\rho \neq 0$. Hence $\rho \equiv 0$, and then also $u \equiv 0$.

Remark 5.1. Theorem 2.1 is valid also in case $\mu = \mu(x)$ provided $\mu(x) \geqslant \mu_0 > 0$ and $\mu \in C^{1+\alpha}(\Gamma)$. The a priori estimates in this case are obtained by arguments similar to used for the variable $Q(x, D)$.

Part 2. The nonlinear problem

6. Localization

The corner points of $\partial\Omega$ are $A = (0, 0)$ and $B = (0, b)$. We parametrize Γ by ω, $\omega \in W$, and designate the position of a point on Γ by $\overline{m}(\omega)$ and the outward normal by $\overline{n}(\omega)$. Let $\bar{l}(\omega)$ be a $C^{k+\alpha}$ $(k \geqslant 6)$ vector field on Γ, transversal to Γ, such that $\bar{l}(\omega) = (0, -1)$ in an ε_0-neighborhood of A and $\bar{l}(\omega) = (0, 1)$ in an ε_0-neighborhood of B. We may take $\bar{l}(\omega) = \overline{n}(\omega)$ outside some ε_1-neighborhood of A and B, such that if γ is sufficiently small then the ω-lines $\overline{m}(\omega) + \eta\bar{l}(\omega)$, $|\eta| < 2\gamma$, do not intersect each other. The mapping $(\omega, \eta) \to y = y(\omega, \eta)$ defined by

$$y = (y_1, y_2) = \overline{m}(\omega) + \eta\bar{l}(\omega) \tag{6.1}$$

is a diffeomorphism from $M = W \times (-\gamma, \gamma)$ onto

$$N = \left\{ y : y = \overline{m}(\omega) + \eta\bar{l}(\omega), \quad (\omega, \eta) \in W \times (-\gamma, \gamma) \right\}.$$

We denote the inverse mapping by Σ; then

$$\Sigma : y \to (\omega(y), \eta(y)) \tag{6.2}$$

maps N onto M.

Let T be any positive number. We assume that the free boundary $\Gamma(t)$ has the form

$$\Gamma(t) = \left\{ (y, t) : y(\omega, t) = \overline{m}(\omega) + \rho(\omega, t)\bar{l}(\omega), \quad t \in [0, T] \right\},$$

where $|\rho(\omega, t)| < \gamma/4$, $\rho(\omega, 0) = 0$. If we set

$$\Phi_\rho(y, t) = \eta(y) - \rho(\omega(y), t), \quad (y, t) \in N \times (0, T)$$

then the free boundary is given by $\Phi_\rho(y, t) = 0$.

Let $\chi(\lambda)$ be a $C_0^\infty(R^1)$ function such that $\chi(\lambda) = 1$ if $|\lambda| < \gamma/3$ and $\chi(\lambda) = 0$ if $|\lambda| > \gamma/2$. We shall use the coordinates (ω, η) to define a diffeomorphism

$$e_\rho : (x, \tau) \to (y, t)$$

from $X_T \equiv R_+^2 \times [0,T]$ onto $Y_T \equiv R_+^2 \times [0,T]$ by setting $t=\tau$ and

$$\omega(y) = \omega(x), \quad \eta(y) = \lambda(x) + \chi\big(\lambda(x)\big)\rho(\omega(x),t) \quad if \quad \big(\omega(x),\lambda(x)\big) \in M,$$
$$y = x \quad otherwise; \tag{6.3}$$

here $\big(\omega(x),\lambda(x)\big)$ is the point $\Sigma(x)$, where Σ is defined as in (6.2).

The transformation e_ρ^{-1} maps $\Omega(t)$ onto $\Omega = \Omega(0)$ and $\Gamma(t)$ onto Γ; the free boundary is given by $e_\rho\big(\{\lambda(x)=0\}\big)$.

We wish to rewrite the Hele–Shaw problem (1.1), (1.7)–(1.9) in terms of the independent variables (x,t), where x varies in the fixed domain Ω. We shall carry out the calculations just near the corner point A, since the analogous computations away from the corner points can be carried out in a similar way (in [2] we have done these calculations in the case $\bar{l}(\omega) = \bar{n}(\omega)$). For simplicity we shall henceforth set $\varphi(x_1) \equiv \varphi_1(x_1)$.

Near A the transformation (6.3) takes the form

$$y_1 = x_1,$$
$$y_2 = x_2 - \chi(z)\rho(x_1,t), \quad z = x_2 - \varphi(x_1). \tag{6.4}$$

Differentiating the second equation in y_2 we find that

$$\frac{\partial x_2}{\partial y_2} = \frac{1}{1-\chi_z\rho} \tag{6.5}$$

and similarly

$$\frac{\partial x_2}{\partial y_1} = -\frac{\chi_z\varphi_{x_1}\rho - \chi\rho_{x_1}}{1-\chi_z\rho}. \tag{6.6}$$

Differentiating once more we get

$$\frac{\partial^2 x_2}{\partial y_2^2} = \frac{\chi_{zz}\rho}{\left(1-\chi_z\rho\right)^2} \tag{6.7}$$

and

$$\frac{\partial^2 x_2}{\partial y_1^2} = -\frac{\chi_{zz}\left(\dfrac{\partial x_2}{\partial y_1} - \dfrac{\partial\varphi}{\partial x_1}\right)\dfrac{\partial\varphi}{\partial x_1}\rho + \chi_z\dfrac{\partial^2\varphi}{\partial x_1^2}\rho}{1-\chi_z\rho}$$

$$-\frac{\chi_z \frac{\partial \varphi}{\partial x_1} \frac{\partial \rho}{\partial x_1} - \chi_z \left(\frac{\partial x_2}{\partial y_1} - \frac{\partial \varphi}{\partial x_1} \right) \frac{\partial \rho}{\partial x_1} - \chi \frac{\partial^2 \rho}{\partial x_1^2}}{1 - \chi_z \rho}$$

$$-\frac{\chi_z \frac{\partial \varphi}{\partial x_1} \rho - \chi \frac{\partial \rho}{\partial x_1}}{\left(1 - \chi_z \rho\right)^2} \left[\chi_{zz} \left(\frac{\partial x_2}{\partial y_1} - \frac{\partial \varphi}{\partial x_1} \right) \rho + \chi_z \frac{\partial \rho}{\partial x_1} \right]. \tag{6.8}$$

Set

$$p\left(y_1, y_2, t\right) = p\left(x_1, y_2\left(x_1, x_2, t\right), t\right) = \upsilon\left(x_1, x_2, t\right). \tag{6.9}$$

Then

$$\begin{aligned}
\frac{\partial p}{\partial y_1} &= \frac{\partial \upsilon}{\partial x_1} + \frac{\partial \upsilon}{\partial x_2} \frac{\partial x_2}{\partial y_1}, \\
\frac{\partial^2 p}{\partial y_1^2} &= \frac{\partial^2 \upsilon}{\partial x_1^2} + 2 \frac{\partial^2 \upsilon}{\partial x_1 \partial x_2} \frac{\partial x_2}{\partial y_1} + \frac{\partial^2 \upsilon}{\partial x_2^2} \left(\frac{\partial x_2}{\partial y_1} \right)^2 + \frac{\partial \upsilon}{\partial x_2} \frac{\partial^2 x_2}{\partial y_1^2}, \\
\frac{\partial p}{\partial y_2} &= \frac{\partial \upsilon}{\partial x_2} \frac{\partial x_2}{\partial y_2}, \\
\frac{\partial^2 p}{\partial y_2^2} &= \frac{\partial^2 \upsilon}{\partial x_2^2} \left(\frac{\partial x_2}{\partial y_2} \right)^2 + \frac{\partial \upsilon}{\partial x_2} \frac{\partial^2 x_2}{\partial y_2^2}.
\end{aligned} \tag{6.10}$$

Hence (1.1) takes the form

$$\begin{aligned}
&\frac{\partial^2 \upsilon}{\partial x_1^2} + 2 \frac{\partial^2 \upsilon}{\partial x_1 \partial x_2} \frac{\partial x_2}{\partial y_1} + \frac{\partial^2 \upsilon}{\partial x_2^2} \left[\left(\frac{\partial x_2}{\partial y_1} \right)^2 + \left(\frac{\partial x_2}{\partial y_2} \right)^2 \right] \\
&\quad + \frac{\partial \upsilon}{\partial x_2} \left(\frac{\partial^2 x_2}{\partial y_1^2} + \frac{\partial^2 x_2}{\partial y_2^2} \right) = 0 \quad in \quad \Omega_T.
\end{aligned} \tag{6.11}$$

The free boundary has the representation

$$\Phi\left(y, t\right) \equiv -y_2 + \varphi\left(y_1\right) - \rho\left(y_1, t\right) = 0.$$

Since

$$\frac{\partial \Phi}{\partial y_1} \frac{dy_1}{dt} + \frac{\partial \Phi}{\partial y_2} \frac{dy_2}{dt} = -\frac{\partial \Phi}{\partial t}, \quad \frac{\nabla_y \Phi}{\left|\nabla_y \Phi\right|} = \overline{n},$$

where $\overline{n}$ is the outward normal, we get

$$V_n = -\frac{\Phi_t}{\left|\nabla_y \Phi\right|} = \frac{\rho_t}{\left|\nabla_y \Phi\right|}.$$

On the other hand,

$$\frac{\partial p}{\partial n} = \nabla_y p \cdot \overline{n} = \frac{1}{\left|\nabla_y \Phi\right|} \nabla_y p \cdot \nabla_y \Phi.$$

Hence the boundary condition (1.8) can be written in the form

$$\rho_t = -\mu \nabla_y p \cdot \nabla_y \Phi.$$

Using (6.5), (6.6), and the relation $\nabla_y \Phi = \left(\varphi_{x_1} - \rho_{x_1}, -1\right)$ on the free boundary, we obtain, after expressing p in terms of υ

$$\rho_t = -\mu \left\{ \left(\frac{\partial \upsilon}{\partial x_1} + \frac{\partial \upsilon}{\partial x_2} \rho_{x_1} \right) \left(\varphi_{x_1} - \rho_{x_1}\right) - \frac{\partial \upsilon}{\partial x_2} \right\}$$

on $x_2 = \varphi(x_1)$. Since

$$\frac{\partial \upsilon}{\partial n} = \frac{\partial \upsilon}{\partial x_1} \frac{\varphi_{x_1}}{\sqrt{1 + \varphi_{x_1}^2}} - \frac{\partial \upsilon}{\partial x_2} \frac{1}{\sqrt{1 + \varphi_{x_1}^2}},$$

we obtain

$$\rho_t = -\mu \left\{ \frac{\partial \upsilon}{\partial n} \sqrt{1 + \varphi_{x_1}^2} + \frac{\partial \rho}{\partial x_1} \left[-\frac{\partial \upsilon}{\partial x_1} + \frac{\partial \upsilon}{\partial x_2} \left(\varphi_{x_1} - \rho_{x_1}\right) \right] \right\} \quad \text{on} \quad \Gamma_T. \tag{6.12}$$

The boundary condition (1.7) can be written in the form

$$\upsilon = \frac{\varphi_{x_1 x_1} - \rho_{x_1 x_1}}{\left[1 + \left(\varphi_{x_1} - \rho_{x_1}\right)^2\right]^{3/2}} \quad \text{on} \quad \Gamma_T \tag{6.13}$$

and (1.9) remains unchanged

$$\frac{\partial \upsilon}{\partial x_1} = g(x_2) \quad \text{on} \quad \Gamma_{0T} \tag{6.14}$$

if γ is sufficiently small compared to a_0 in (1.10).

In conclusion:

Lemma 6.1. *In some neighborhood* V_1 *of the corner point* A *the system* (1.1), (1.7)–(1.9) *takes the form* (6.11)–(6.14) *in the variables* (x_1, x_2, t).

7. A perturbation form of the system near a corner point

In this section, we rewrite the system (6.11)–(6.14) as a system $Az = F(z)$ where A is a linear operator as in the linear problem and $F(z)$ is a nonlinear perturbation. The linearization will be done about the initial data and our purpose is to get explicit formulas near the corner points.

Consider the function $\upsilon_0(x) = \upsilon(x, 0)$. Since $x = y$ at $t = 0$, $\upsilon_0(x)$ satisfies the conditions

$$\Delta\upsilon_0 = 0 \quad \text{in } \Omega, \quad \upsilon_0 = \kappa(\Gamma) \quad \text{on} \quad \Gamma, \quad \frac{\partial\upsilon_0}{\partial x_1} = g \quad \text{on} \quad \Gamma_0,$$

where $\kappa(\Gamma)$ is the curvature of Γ. We will assume that there exist such neighborhoods V_1 and V_2 of the points A and B that $\varphi_{j,x_1x_1} \in E^{5+\alpha}_{5+\alpha}\left(\Gamma \cap V_j\right)$, $j = 1, 2$. Analogously to (3.5) we have

$$\|\upsilon_0\|_{E^{5+\alpha}_{5+\alpha}} \leqslant const.\ \|\kappa(\Gamma)\|_{E^{5+\alpha}_{5+\alpha}(\Gamma)}, \tag{7.1}$$

indeed, the proof is similar but much simpler since $\rho(x_1, 0) \equiv 0$. In this proof we need to make assumption (2.5) and to apply Theorems 6.4.1.1 and 6.4.1.3 of [8].

From (6.12) we see that

$$\frac{\partial\rho}{\partial t}(x_1, 0) = -\mu\frac{\partial\upsilon_0}{\partial n}\sqrt{1+\varphi^2_{x_1}}. \tag{7.2}$$

Let

$$m(x_1, t) = t\, m_0(x_1), \quad \text{where } m_0(x_1) = -\mu\frac{\partial\upsilon_0}{\partial n}\sqrt{1+\varphi^2_{x_1}} \tag{7.3}$$

and introduce the functions

$$w(x, t) = \upsilon(x, t) - \upsilon_0(x), \tag{7.4}$$

$$\sigma(x_1, t) = \rho(x_1, t) - m(x_1, t). \tag{7.5}$$

We shall now rewrite the system (6.11)–(6.14) in terms of the functions $w\,(x,t)$, $\sigma\,(x_1,t)$. From (6.11) we get

$$\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}=F_0\,(w,\sigma)\quad\text{in}\quad(\Omega\cap V_1)_T\,,\tag{7.6}$$

where

$$\begin{aligned}-\,F_0\,(w,\sigma)\;=\;&\frac{\partial^2 \upsilon_0}{\partial x_1^2}+2\frac{\partial^2\,(w+\upsilon_0)}{\partial x_1\partial x_2}\,\frac{\partial x_2}{\partial y_1}+\frac{\partial^2 w}{\partial x_2^2}\left[-1+\left(\frac{\partial x_2}{\partial y_1}\right)^2+\left(\frac{\partial x_2}{\partial y_2}\right)^2\right]\\&+\frac{\partial^2 \upsilon_0}{\partial x_2^2}\left[\left(\frac{\partial x_2}{\partial y_1}\right)^2+\left(\frac{\partial x_2}{\partial y_2}\right)^2\right]+\frac{\partial\,(w+\upsilon_0)}{\partial x_2}\left[\frac{\partial^2 x_2}{\partial y_1^2}+\frac{\partial^2 x_2}{\partial y_2^2}\right]\end{aligned}\tag{7.7}$$

and $\partial x_i/\partial y_j$, $\partial^2 x_i/\partial y_j^2$ are given by (6.5)–(6.8) and are considered to depend on σ through the relation (7.5).

Next we rewrite (6.12) in the form

$$\frac{\partial\sigma}{\partial t}+\mu\,\frac{\partial w}{\partial n}\sqrt{1+\varphi_{x_1}^2}=F\,(w,\sigma)\quad\text{on}\quad(\Gamma\cap V_1)_T\,,\tag{7.8}$$

where

$$\begin{aligned}-F\,(w,\sigma)\;=\;&\frac{\partial m}{\partial t}+\mu\left\{\frac{\partial\upsilon_0}{\partial n}\sqrt{1+\varphi_{x_1}^2}-\left(\frac{\partial m}{\partial x_1}+\frac{\partial\sigma}{\partial x_1}\right)\left(\frac{\partial w}{\partial x_1}+\frac{\partial\upsilon_0}{\partial x_1}\right)\right\}\\&-\mu\left(\frac{\partial m}{\partial x_1}+\frac{\partial\sigma}{\partial x_1}\right)\left(\frac{\partial w}{\partial x_2}+\frac{\partial\upsilon_0}{\partial x_2}\right)\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial m}{\partial x_1}-\frac{\partial\sigma}{\partial x_1}\right).\end{aligned}\tag{7.9}$$

From (6.13) we have

$$w+\frac{\partial}{\partial\omega}\left(\frac{1}{\left(1+\varphi_{x_1}^2\right)^{1/2}}\frac{\partial\sigma}{\partial\omega}\right)=F_1\,(\sigma)\quad\text{on}\quad(\Gamma\cap V_1)_T\,,\tag{7.10}$$

where ω is the arclength parameter and

$$\begin{aligned}F_1\,(\sigma)\;=\;&-\upsilon_0+\frac{\varphi_{x_1x_1}-m_{x_1x_1}}{\left[1+\left(\varphi_{x_1}-\sigma_{x_1}-m_{x_1}\right)^2\right]^{3/2}}-\frac{2\sigma_{x_1}\varphi_{x_1}\varphi_{x_1x_1}}{\left(1+\varphi_{x_1}^2\right)^{5/2}}\\&-\sigma_{x_1x_1}\left\{\frac{1}{\left[1+\left(\varphi_{x_1}-\sigma_{x_1}-m_{x_1}\right)^2\right]^{3/2}}-\frac{1}{\left[1+\varphi_{x_1}^2\right]^{3/2}}\right\}.\end{aligned}\tag{7.11}$$

Finally, (6.14) takes the form

$$\frac{\partial w}{\partial x_1} = 0 \quad \text{on} \quad (\Gamma_0 \cap V_1)_T \, . \tag{7.12}$$

8. Auxiliary lemmas

In Section 9, we shall prove that the perturbation system $Az = F(z)$ has a unique solution. To do this we need some calculus type lemmas.

Lemma 8.1. *Let D denote bounded domain either Ω or Γ. If f, g are functions from $E_s^{k+\alpha,\beta,\alpha}(D_T)$ then*

$$\|fg\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \leqslant const.\, \|f\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, \|g\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, . \tag{8.1}$$

Proof. Consider the case $k = 0$. To estimate the norm of the product we use the simple inequalities:

$$\sup_{\overline{D}_T} R^{-s} \, |fg| \leqslant \left(\sup_{\overline{D}_T} R^{-s} \, |f| \right) \left(\sup_{\overline{D}_T} |g| \right) \leqslant c \, \|f\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, \|g\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, ,$$

$$\begin{aligned} \langle fg \rangle^{(\alpha)}_{x,s,D_T} \; &\leqslant \; \langle f \rangle^{(\alpha)}_{x,s,D_T} \sup_{\overline{D}_T} |g| + \langle g \rangle^{(\alpha)}_{x,s,D_T} \sup_{\overline{D}_T} |f| \\ &\leqslant \; c \, \|f\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, \|g\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, , \end{aligned} \tag{8.2}$$

$$\begin{aligned} [fg]^{(\alpha,\beta)}_{s,D_T} \leqslant & [f]^{(\alpha,\beta)}_{s,D_T} \sup_{\overline{D}_T} |g| + [g]^{(\alpha,\beta)}_{s,D_T} \sup_{\overline{D}_T} |f| \\ & + \langle f \rangle^{(\alpha)}_{x,s,D_T} \, \langle g \rangle^{(\beta)}_{t,s,D_T} + \langle g \rangle^{(\alpha)}_{x,s,D_T} \, \langle f \rangle^{(\beta)}_{t,s,D_T} \leqslant c \, \|f\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, \|g\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, . \end{aligned}$$

From here the estimate (8.1) follows for $k = 0$. The case $k \geqslant 1$ is considered by successive application of (8.2) and the inequality

$$\|f\|_{E_{s-l}^{k-l+\alpha,\beta,\alpha}(D_T)} \leqslant \|f\|_{E_s^{k+\alpha,\beta,\alpha}(D_T)} \, , \quad l \leqslant k. \tag{8.3}$$

Lemma 8.2. *If $v \in N^{4+\alpha}_{1+\alpha}(\Gamma_T)$ and $v(x,0) = 0$, then, for $T \leqslant 1$, the following inequality holds*:

$$\|v_{xx}\|_{E_\alpha^{\alpha,\alpha/3,\alpha}(\Gamma_T)} \leqslant const. T^{1/3} \, \|v\|_{N^{4+\alpha}_{1+\alpha}(\Gamma_T)} \, . \tag{8.4}$$

Proof. We begin with

$$\sup_{\Gamma_T} R^{-\alpha}|v_{xx}| \leqslant \sup_{\Gamma_T} R^{-\alpha} t^{\frac{2+\alpha}{3}} \left|\frac{v_{xx}(x,t)-v_{xx}(x,0)}{t^{\frac{2+\alpha}{3}}}\right| \leqslant cT^{\frac{2+\alpha}{3}} \left\langle D_x^2 v\right\rangle_{t,2+\alpha,\Gamma_T}^{\left(\frac{2+\alpha}{3}\right)}$$

$$\leqslant cT^{\frac{2+\alpha}{3}} \left\|D_x^2 v\right\|_{C_{2+\alpha}^{\alpha,\frac{2+\alpha}{3}}(\Gamma_T)} \leqslant cT^{\frac{2+\alpha}{3}} \|v\|_{N_{1+\alpha}^{4+\alpha}(\Gamma_T)}. \tag{8.5}$$

Since

$$\sup_{\Gamma_T}|v_{xxx}| \leqslant cT^{\frac{1+\alpha}{3}} \left\langle D_x^3 v\right\rangle_{t,3+\alpha,\Gamma_T}^{\left(\frac{1+\alpha}{3}\right)} \leqslant cT^{\frac{1+\alpha}{3}} \|v\|_{N_{1+\alpha}^{4+\alpha}(\Gamma_T)},$$

we get

$$\sup_{x,y,t\in\Gamma_T} R^{-\alpha+\alpha}(x,y) \frac{\left|v_{xx}(x,t)-v_{yy}(y,t)\right|}{|x-y|^{\alpha}} \leqslant \sup_{x,y,t\in\Gamma_T} |x-y|^{1-\alpha}|v_{xxx}|$$

$$\leqslant c \sup_{x,y,t\in\Gamma_T}|v_{xxx}| \leqslant cT^{\frac{1+\alpha}{3}} \|v\|_{N_{1+\alpha}^{4+\alpha}(\Gamma_T)}. \tag{8.6}$$

Next

$$\sup_{x,y,t\in\Gamma_T} R^{-\alpha} \frac{|v_{xx}(x,t)-v_{xx}(x,\tau)|}{|t-\tau|^{\alpha/3}} \leqslant cT^{\frac{2}{3}} \langle v_{xx}\rangle_{t,2+\alpha,\Gamma_T}^{\left(\frac{2+\alpha}{3}\right)} \leqslant cT^{\frac{2}{3}} \|v\|_{N_{1+\alpha}^{4+\alpha}(\Gamma_T)}, \tag{8.7}$$

$$\sup_{x,y,t,\tau\in\Gamma_T} R^{-\alpha+\alpha}(x,y) \frac{\left|v_{xx}(x,t)-v_{yy}(y,t)-v_{xx}(x,\tau)+v_{yy}(y,\tau)\right|}{|x-y|^{\alpha}|t-\tau|^{\alpha/3}}$$

$$\leqslant \sup_{x,y,t,\tau\in\Gamma_T} |x-y|^{1-\alpha}|t-\tau|^{1/3} \langle v_{xxx}\rangle_{t,3+\alpha,\Gamma_T}^{\left(\frac{1+\alpha}{3}\right)} \leqslant cT^{1/3} \langle v_{xxx}\rangle_{t,3+\alpha,\Gamma_T}^{\left(\frac{1+\alpha}{3}\right)}$$

$$\leqslant cT^{\frac{1}{3}} \|v\|_{N_{1+\alpha}^{4+\alpha}(\Gamma_T)}. \tag{8.8}$$

The inequalities (8.5)–(8.8) yield the assertion of the lemma.

Lemma 8.3. *Let* $v \in C_{s+2}^{\alpha,\frac{1+\alpha}{3}}(\Gamma_T) \cap E_{s+1}^{1+\alpha,\alpha/3,\alpha}(\Gamma_T)$. *Then*

$$[v]_{\alpha,\Gamma_T}^{(\alpha,\alpha/3)} \leqslant const.T^{\frac{1-\alpha}{3}} \left(\|v\|_{C_{s+2}^{\alpha,\frac{1+\alpha}{3}}(\Gamma_T)} + \|v\|_{E_{s+1}^{1+\alpha,\alpha/3,\alpha}(\Gamma_T)}\right). \tag{8.9}$$

Proof. First consider the case $|x-y| \leqslant |t-\tau|^{1/3}$. Then

$$|v(x,t)-v(y,t)-v(x,\tau)+v(y,\tau)| \leqslant \int_y^x \left|\frac{\partial v}{\partial z}(z,t)-\frac{\partial v}{\partial z}(z,\tau)\right| dz$$

$$\leqslant c\int_y^x |t-\tau|^{\alpha/3}\langle v_z\rangle_{t,s,\Gamma_T}^{(\alpha/3)}\, dz \leqslant c\,|x-y|\,|t-\tau|^{\alpha/3}\langle v_z\rangle_{t,s,\Gamma_T}^{(\alpha/3)}.$$

Hence,

$$\frac{|v(x,t)-v(y,t)-v(x,\tau)+v(y,\tau)|}{|x-y|^{\alpha}|t-\tau|^{\alpha/3}} \leqslant c\,|x-y|^{1-\alpha}\langle v_z\rangle_{t,s,\Gamma_T}^{(\alpha/3)}$$
$$\leqslant c\,|t-\tau|^{(1-\alpha)/3}\langle v_z\rangle_{t,s,\Gamma_T}^{(\alpha/3)} \leqslant cT^{(1-\alpha)/3}\|v\|_{E_{s+1}^{1+\alpha,\alpha/3,\alpha}(\Gamma_T)}. \tag{8.10}$$

Now let $|x-y| \geqslant |t-\tau|^{1/3}$. Since

$$|v(x,t)-v(y,t)-v(x,\tau)+v(y,\tau)| \leqslant |v(x,t)-v(x,\tau)|+|v(y,t)-v(y,\tau)|$$

we have

$$\frac{|v(x,t)-v(y,t)-v(x,\tau)+v(y,\tau)|}{|x-y|^{\alpha}|t-\tau|^{\alpha/3}} \leqslant c\,\frac{\langle v\rangle_{t,s+2,\Gamma_T}^{(1+\alpha)/3}\,|t-\tau|^{(1+\alpha)/3}}{|t-\tau|^{2\alpha/3}}$$
$$\leqslant c\,\langle v\rangle_{t,s+2,\Gamma_T}^{(1+\alpha)/3}\,|t-\tau|^{(1-\alpha)/3} \leqslant cT^{(1-\alpha)/3}\|v\|_{C_{s+2}^{\alpha,\frac{1+\alpha}{3}}(\Gamma_T)}. \tag{8.11}$$

Combining the inequalities (8.10) and (8.11), we obtain the assertion (8.8).

9. Proof of the main result

Since the transformation $e_\rho^{-1}:(y,t)\to(x,t)$ is defined for all $y\in\Omega(t)$, $t\in(0,T)$, under the condition that $\rho(x,t)$ is a smooth function and $\rho(x,0)=0$ we have reduced our free boundary problem to a perturbation problem of the form (see also [2])

$$\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2} = F_0(w,\sigma) \quad \text{in } \Omega_T, \tag{9.1}$$

$$\frac{\partial\sigma}{\partial t}+\mu q(\omega)\frac{\partial w}{\partial n} = F(w,\sigma) \quad \text{on } \Gamma_T, \tag{9.2}$$

$$w+\frac{\partial}{\partial\omega}\left(\frac{\sigma_\omega}{q(\omega)}\right) = F_1(\sigma) \quad \text{on } \Gamma_T, \tag{9.3}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \ \ \Gamma_{0T}, \tag{9.4}$$

$$\sigma(\omega, 0) = 0, \quad \sigma(A, t) = \sigma(B, t) = 0, \tag{9.5}$$

where $q(\omega)$ is a given function. In a neighborhood of the corner point A $q(\omega) = \sqrt{1+\varphi_{x_1}^2}$ and the vector-function $\Im(w,\sigma) = (F_0(w,\sigma), \ F(w,\sigma), \ F_1(\sigma))$ is defined by (7.7), (7.9), (7.11), where the equation $l(\omega) = (0,-1)$ was used. Away from neighborhoods of the angular points A and B the function $\Im(w,\sigma)$ can be calculated similarly to what has been done in [2].

The problem (9.1)–(9.5) has the form of the linear system (2.1) with $f_0 = F_0$, $f = F$, $f_1 = F_1$, $f_2 = 0$ in the fixed domain Ω_T, but these functions depend on a solution w, σ. Let $\psi = (w,\sigma)$, $\Im = (F_0, F, F_1)$ and introduce, correspondingly, the function spaces H_D and H_R (so that $\psi \in H_D$, $\Im \in H_R$) by

$$H_D = E_{s+1}^{2+\alpha,\alpha/3,\alpha}(\Omega_T) \times N_s^{4+\alpha}(\Gamma_T),$$

$$H_R = E_{s-1}^{\alpha,\alpha/3,\alpha}(\Omega_T) \times E_s^{1+\alpha,\alpha/3,\alpha}(\Gamma_T) \times E_{s+1}^{2+\alpha,\alpha/3,\alpha}(\Gamma_T).$$

We will assume from now on that $s = 1+\alpha$. If we write the system (9.1)– (9.5) briefly in the form

$$A\psi = \Im(\psi) \tag{9.6}$$

then, by Theorem 2.1, A has a bounded inverse A^{-1}, so that

$$\psi = A^{-1}\Im(\psi) \equiv S(\psi) \tag{9.7}$$

and S is a nonlinear operator. We will show that S is a contraction operator.

Let $B_r(\psi) = \left\{\psi \in H_D : \|\psi\|_{H_D} \leqslant r, \ \ \psi(x,0) = 0\right\}$, $r \leqslant r_0$, be a ball of radius r in the space H_D, centered at the origin, for some r_0, to be determined later on.

Lemma 9.1. *The following inequalities hold for the right-hand side of the problem* (9.6):

$$\|\Im(\psi_1) - \Im(\psi_2)\|_{H_R} \leqslant g(r,T)\,\|\psi_1 - \psi_2\|_{H_D}, \quad \psi_1, \psi_2 \in B_r, \tag{9.8}$$

$$\|\Im(0)\|_{H_R} \leqslant \delta(T) \tag{9.9}$$

with $\delta(T) \to 0$, $g(r,T) \to 0$ *as* $r \to 0$, $T \to 0$.

Proof. We begin by introducing a finite open cover $\left\{V_j\right\}_{j=1}^{n_0}$ of $\overline{\Omega}$ with the following requirements: V_1 is a neighborhood of A and V_2 is a neighborhood of B; V_3 is a neighborhood of $\Gamma_0 \backslash \left(\overline{\Gamma}_0 \cap \left(\overline{V}_1 \cup \overline{V}_2\right)\right)$ which stays away from Γ; V_4 is an open subset of Ω, and V_j for $5 \leqslant j \leqslant n_0$ cover $\Gamma \backslash (\Gamma \cap (V_1 \cup V_2))$. Let $\left\{\varsigma_j\right\}$ be a partition of unity subordinate to the covering $\left\{V_j\right\}$. Observe that if $u = \sum_{j=1}^{n_0} \varsigma_j u \equiv \sum_{j=1}^{n_0} u_j$ then $\|u\| \leqslant \sum_{j=1}^{n_0} \left\|u_j\right\| \leqslant n_0 \sup\limits_j \left\|u_j\right\|$.

To prove (9.9) we consider, as an example, $\|\varsigma_1 \Im(0)\|_{H_R}$ and estimate it using the expressions given by (7.7), (7.9), (7.11). We begin with $\varsigma_1 F_0(0,0)$. From the definition in (7.7) we have, since $\Delta v_0 = 0$,

$$-F_0(0,0) = 2\frac{\partial^2 v_0}{\partial x_1 \partial x_2}\frac{\partial x_2}{\partial y_1}\bigg|_{\sigma=0} + \frac{\partial^2 v_0}{\partial x_2^2}\left[\left(\frac{\partial x_2}{\partial y_1}\right)^2 + \left(\frac{\partial x_2}{\partial y_2}\right)^2 - 1\right]\bigg|_{\sigma=0} + \frac{\partial v_0}{\partial x_2}\left[\frac{\partial^2 x_2}{\partial y_1^2} + \frac{\partial^2 x_2}{\partial y_2^2}\right]\bigg|_{\sigma=0}. \tag{9.10}$$

From (6.6) we get

$$\frac{\partial x_2}{\partial y_1}\bigg|_{\sigma=0} = -\frac{\chi_z \varphi_{x_1} m - \chi m_{x_1}}{1 - \chi_z m} = t\left\{-\frac{\chi_z \varphi_{x_1} m_0 - \chi m_{0x_1}}{1 - \chi_z m}\right\} \equiv t P_1(x,t), \tag{9.11}$$

where $P_1(x,t) \in E_{3+\alpha}^{3+\alpha,\alpha/3,\alpha}(\Omega_T)$ for $t \cdot \max\limits_x \left|\chi_z m_0\right| \leqslant \frac{1}{2}$. From (6.5) it follows

$$\frac{\partial x_2}{\partial y_2}\bigg|_{\sigma=0} = \frac{1}{1-\chi_z m} \in C^{4+\alpha}(\Omega_T),$$

$$\left(\frac{\partial x_2}{\partial y_2}\right)^2\bigg|_{\sigma=0} - 1 = t\,\frac{2\chi_z m_0 - \chi_z^2 m_0^2 t}{\left(1-\chi_z m\right)^2} \equiv t P_2(x,t),\ P_2(x,t) \in E_{4+\alpha}^{4+\alpha,\alpha/3,\alpha}(\Omega_T). \tag{9.12}$$

Next

$$\frac{\partial^2 x_2}{\partial y_2^2}\bigg|_{\sigma=0} = t\,\frac{\chi_{zz} m_0}{\left(1-\chi_z m\right)^2} \equiv t P_3(x,t), \quad P_3(x,t) \in E_{4+\alpha}^{4+\alpha,\alpha/3,\alpha}(\Omega_T). \tag{9.13}$$

From (6.8) we can also see that the "worst" term in the representation of $\left.\frac{\partial^2 x_2}{\partial y_1^2}\right|_{\sigma=0}$ is $t \cdot \chi m_{0x_1x_1} / \left(1 - \chi_z m\right)$, so that we get

$$\left.\frac{\partial^2 x_2}{\partial y_1^2}\right|_{\sigma=0} = t \cdot P_4(x,t), \quad P_4(x,t) \in E_{2+\alpha}^{2+\alpha,\alpha/3,\alpha}(\Omega_T). \tag{9.14}$$

Combining (9.11)–(9.14) and recalling that $v_0 \in E_{5+\alpha}^{5+\alpha}(\Omega)$, we can write: $\varsigma_1 F_0(0,\ 0)(x,t) = t \cdot Q_0(x,t)$, $Q_0(x,t) \in E_{2+\alpha}^{2+\alpha,\alpha/3,\alpha}(\Omega_T)$. Direct calculations give

$$\|\varsigma_1 F_0(0,0)(x,t)\|_{E_\alpha^{\alpha,\alpha/3,\alpha}(\Omega_T)} \leqslant const.T^{1-\alpha/3}\|Q_0\|_{E_\alpha^{\alpha,\alpha/3,\alpha}(\Omega_T)} \leqslant const.T^{1-\alpha/3}. \tag{9.15}$$

From (7.9) and the equality $m_0(x) + \frac{\partial v_0}{\partial n}\left(1+\varphi_{x_1}\right)^{1/2} = 0$ we get

$$-F(0,0)(x_1,t) = -\mu \frac{\partial m}{\partial x_1}\left[\frac{\partial v_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2}\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial m}{\partial x_1}\right)\right]$$

and one can see that $\varsigma_1 F(0,0)(x_1,t) = t \cdot Q(x_1,t)$, $Q(x_1,t) \in E_{4+\alpha}^{4+\alpha,\alpha/3,\alpha}(\Gamma_T)$ and therefore analogously to (9.15)

$$\|\varsigma_1 F(0,0)(x_1,t)\|_{E_{1+\alpha}^{1+\alpha,\alpha/3,\alpha}(\Gamma_T)} \leqslant const.T^{1-\alpha/3}. \tag{9.16}$$

From (7.11)

$$F_1(0)(x_1,t) = -v_0 + \frac{\varphi_{x_1x_1} - m_{x_1x_1}}{\left(1+\left(\varphi_{x_1} - m_{x_1}\right)^2\right)^{3/2}}.$$

Since $m(x_1,0) = 0$ and $v_0(x) = \kappa(\Gamma)$ on Γ, we have

$$F_1(0)(x_1,0) = -v_0 + \frac{\varphi_{x_1x_1}}{\left(1+\varphi_{x_1}^2\right)^{3/2}} = 0.$$

Hence, we can write

$$\begin{aligned}\varsigma_1 F_1(0)(x_1,t) &= \varsigma_1 F_1(0)(x_1,t) - \varsigma_1 F_1(0)(x_1,0)\\ &= \varsigma_1 \varphi_{x_1x_1}\left[\frac{1}{\left(1+\left(\varphi_{x_1} - m_{x_1}\right)^2\right)^{3/2}} - \frac{1}{\left(1+\varphi_{x_1}^2\right)^{3/2}}\right]\\ &\quad - \frac{\varsigma_1 m_{x_1x_1}}{\left(1+\left(\varphi_{x_1} - m_{x_1}\right)^2\right)^{3/2}}.\end{aligned} \tag{9.17}$$

Note that each of the two terms on the right-hand side of (9.17) is equal to zero for $t = 0$. We first estimate

$$\left\| \varsigma_1 \frac{m_{x_1 x_1}}{\left(1 + \left(\varphi_{x_1} - m_{x_1}\right)^2\right)^{3/2}} \right\|_{E_{2+\alpha}^{2+\alpha,\alpha/3,\alpha}(\Gamma_T)}. \tag{9.18}$$

Here we shall need the assumption that $\|\kappa(\Gamma)\|_{E_{5+\alpha}^{5+\alpha}(\Gamma)} \leqslant const$. The "worst" term to be estimated in (9.18) is

$$\left\| \varsigma_1 \frac{t \cdot D_{x_1}^4 m_0}{\left(1 + \left(\varphi_{x_1} - t m_{0x_1}\right)^2\right)^{3/2}} \right\|_{E_{\alpha}^{\alpha,\alpha/3,\alpha}(\Gamma_T)}$$

and we can bound it as follows:

$$\left\| \varsigma_1 \frac{t \cdot D_{x_1}^4 m_0}{\left(1 + \left(\varphi_{x_1} - t m_{0x_1}\right)^2\right)^{3/2}} \right\|_{E_{\alpha}^{\alpha,\alpha/3,\alpha}(\Gamma_T)} \leqslant const.T^{1-\alpha/3}. \tag{9.19}$$

To estimate the first term in (9.17), we represent it by the integral mean value theorem in the form $t \cdot m_{0x_1} P_5(x_1, t)$ where

$$P_5(x_1, t) = 3\varsigma_1 \varphi_{x_1 x_1} \int_0^1 \frac{\left(\varphi_{x_1} - s m_{x_1}\right) ds}{\left(1 + \left(\varphi_{x_1} - s m_{x_1}\right)^2\right)^{5/2}} \in C^{3+\alpha,\alpha/3,\alpha}(\Gamma_T)$$

from which we obtain the bound

$$\left\| t \cdot m_{0x_1} P_5(x_1, t) \right\|_{E_{2+\alpha}^{2+\alpha,\alpha/3,\alpha}(\Gamma_T)} \leqslant const.T^{1-\alpha/3}.$$

Combining this with (9.19), we get

$$\|\varsigma_1 F_1(0)(x_1, t)\|_{E_{2+\alpha}^{2+\alpha,\alpha/3,\alpha}(\Gamma_T)} \leqslant const.T^{1-\alpha/3}. \tag{9.20}$$

The estimates (9.15), (9.16), and (9.20) together with similar estimates for $\varsigma_j \Im(0)(x, t)$, $j = 2, \ldots, n_0$, prove the inequality (9.9) in Lemma 9.1.

To prove the inequality (9.8) we note that the function $\Im\left(\psi\right)$ is analytic in ψ and its derivatives and it can be represented in the form $\Im\left(\psi\right)=\Im\left(0\right)+\Lambda\left(\psi\right)+\ O\left(\psi^{2}\right)$, where $\Lambda\left(\psi\right)$ is linear in ψ and its derivatives, and $O\left(\psi^{2}\right)$ denotes the terms of higher order in ψ and its derivatives. We begin by estimating the linear terms in the difference $\Im\left(\psi_1\right)-\Im\left(\psi_2\right)$.

We denote by $L_0\left(\psi\right)$ all linear terms in $\varsigma_1 F_0\left(w,\sigma\right)$. The "worst" term comes from $\varsigma_1 \frac{\partial v_0}{\partial x_2}\frac{\partial^2 x_2}{\partial y_1^2}\Big|_{\sigma=0}$, and it is

$$\varsigma_1 \frac{\partial v_0}{\partial x_2}\frac{\chi\left(z\right)\sigma_{x_1x_1}}{1-\chi_z m}=\sigma_{x_1x_1}P_6\left(x,t\right),\quad P_6\left(x,t\right)\in E^{4+\alpha,\alpha/3,\alpha}_{4+\alpha}\left(\Omega_T\right).$$

We want to prove that

$$\left\|\left(\sigma_{1,x_1x_1}-\sigma_{2,x_1x_1}\right)P_6\left(x,t\right)\right\|_{E^{\alpha,\alpha/3,\alpha}_{\alpha}\left(\Omega_T\right)}\leqslant\delta\left(T\right)\left\|\sigma_1-\sigma_2\right\|_{N^{4+\alpha}_{1+\alpha}\left(\Gamma_T\right)}. \tag{9.21}$$

Set $v\left(x_1,t\right)=\sigma_{1,x_1x_1}-\sigma_{2,x_1x_1}$. Since $v\left(x_1,0\right)=0$, we can apply Lemmas 8.1 and 8.2 to v and thus obtain (9.21) with $\delta\left(T\right)=cT^{1/3}$. The other terms in $L_0\left(\psi_1\right)-L_0\left(\psi_2\right)$ can be estimated more easily, using the same Lemmas 8.1 and 8.2, so that

$$\left\|L_0\left(\psi_1\right)-L_0\left(\psi_2\right)\right\|_{E^{\alpha,\alpha/3,\alpha}_{\alpha}\left(\Omega_T\right)}\leqslant\delta\left(T\right)\left\|\psi_1-\psi_2\right\|_{H_D}, \tag{9.22}$$

where $\delta\left(T\right)\to 0$ as $\delta\to 0$.

The linear terms in $\varsigma_1 F\left(w,\sigma\right)$ are contained in

$$\begin{aligned}L\left(\psi\right)\ =\ &\varsigma_1\mu t\cdot m_{0x_1}\left[\frac{\partial w}{\partial x_1}+\frac{\partial w}{\partial x_2}\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial m}{\partial x_1}\right)-\frac{\partial v_0}{\partial x_2}\frac{\partial\sigma}{\partial x_1}\right]\\&+\varsigma_1\mu\frac{\partial\sigma}{\partial x_1}\left[\frac{\partial v_0}{\partial x_1}+\frac{\partial v_0}{\partial x_2}\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial m}{\partial x_1}\right)\right].\end{aligned}$$

The first term includes the factor t and the second term includes the factor σ_{x_1}. Since the $E^{1+\alpha,\alpha/3,\alpha}_{1+\alpha}\left(\Gamma_T\right)$-norm of σ_{x_1} is bounded by $\delta\left(T\right)$, we get

$$\left\|L\left(\psi_1\right)-L\left(\psi_2\right)\right\|_{E^{1+\alpha,\alpha/3,\alpha}_{1+\alpha}\left(\Gamma_T\right)}\leqslant\delta\left(T\right)\left\|\psi_1-\psi_2\right\|_{H_D}. \tag{9.23}$$

The linear terms in $\varsigma_1 F_1(\sigma)$ have the form

$$L_1(\psi) = \varsigma_1 \sigma_{x_1} \left[-\frac{2\varphi_{x_1}\varphi_{x_1 x_1}}{\left(1+\varphi_{x_1}^2\right)^{5/2}} + \frac{3\left(\varphi_{x_1} - m_{x_1}\right)}{\left(1+\left(\varphi_{x_1}-m_{x_1}\right)^2\right)^{5/2}} \right] - \varsigma_1 \sigma_{x_1 x_1} \left[\frac{1}{\left(1+\left(\varphi_{x_1}-m_{x_1}\right)^2\right)^{3/2}} - \frac{1}{\left(1+\varphi_{x_1}^2\right)^{3/2}} \right]. \tag{9.24}$$

The last term in (9.24) is proportional to tm_{0x_1}, so that the "worst" term in $L_1(\psi_1) - L_1(\psi_2)$ has the form $\left(\sigma_{1x_1} - \sigma_{2x_1}\right) P_7(x_1, t)$ with $P_7(x_1, t) \in E_{3+\alpha}^{3+\alpha,\alpha/3,\alpha}(\Gamma_T)$.

Let $v = D_{x_1}^3 \sigma_1 - D_{x_1}^3 \sigma_2$. To estimate $\left(\sigma_{1x_1} - \sigma_{2x_1}\right) P_7(x_1, t)$ in the norm of $E_{2+\alpha}^{2+\alpha,\alpha/3,\alpha}(\Gamma_T)$ it suffices to calculate the higher seminorms in $\|v\|_{E_\alpha^{\alpha,\alpha/3,\alpha}(\Gamma_T)}$. The seminorm $[v]_{\alpha,\Gamma_T}^{(\alpha,\alpha/3)}$ is estimated by Lemma 8.3, so that

$$[v]_{\alpha,\Gamma_T}^{(\alpha,\alpha/3)} \leqslant cT^{(1-\alpha)/3} \|\sigma_1 - \sigma_2\|_{N_{1+\alpha}^{4+\alpha}(\Gamma_T)}. \tag{9.25}$$

Next, since $v(x, 0) = 0$ we have, using the definition of the seminorm $[v]_{\alpha,\Gamma_T}^{(\alpha,\alpha/3)}$,

$$\langle v\rangle_{x,\alpha,\Gamma_T}^{(\alpha)} = \sup_{x,y,t} \frac{|v(x,t) - v(y,t)|}{|x-y|^\alpha} \leqslant T^{\alpha/3} [v]_{\alpha,\Gamma_T}^{(\alpha,\alpha/3)} \tag{9.26}$$

and, furthermore,

$$\begin{aligned} \langle v\rangle_{t,\alpha,\Gamma_T}^{(\alpha/3)} &= \sup_{x,t,\tau} r^{-\alpha} \frac{|v(x,t) - v(x,\tau)|}{|t-\tau|^{\alpha/3}} \leqslant cT^{1/3} \langle v\rangle_{3+\alpha,\Gamma_T}^{((1+\alpha)/3)} \\ &\leqslant cT^{1/3} \|\sigma_1 - \sigma_2\|_{N_{1+\alpha}^{4+\alpha}(\Gamma_T)}. \end{aligned} \tag{9.27}$$

Combining the inequalities (9.25)–(9.27) we get

$$\left\|L_1(\psi_1) - L_1(\psi_2)\right\|_{E_{2+\alpha}^{2+\alpha,\alpha/3,\alpha}(\Gamma_T)} \leqslant \delta(T) \left\|\psi_1 - \psi_2\right\|_{H_D}. \tag{9.28}$$

The value $\left\|\varsigma_1\left(\Lambda(\psi_1) - \Lambda(\psi_2)\right)\right\|_{H_R}$ is bounded by $\delta(T)\left\|\psi_1 - \psi_2\right\|_{H_D}$ due to the inequalities (9.22), (9.23), (9.28). Similar estimates hold for $j = 2, 3, \ldots, n_0$ so that

$$\left\|\Lambda(\psi_1) - \Lambda(\psi_2)\right\|_{H_R} \leqslant \delta(T) \left\|\psi_1 - \psi_2\right\|_{H_D}, \quad \delta(T) \to 0 \ \ as \ \ T \to 0. \tag{9.29}$$

Denote by $\Phi(\psi) = \Im(\psi) - \Im(0) - \Lambda(\psi)$ the nonlinear part of $\Im(\psi)$. Since $\Phi(\psi)$ is analytic in ψ

$$\begin{aligned}\left\|\Phi(\psi_1) - \Phi(\psi_2)\right\|_{H_R} &\leqslant const.\left(\left\|\psi_1\right\|_{H_D} + \left\|\psi_2\right\|_{H_D}\right)\left\|\psi_1 - \psi_2\right\|_{H_D} \\ &\leqslant const.\cdot r_0 \left\|\psi_1 - \psi_2\right\|_{H_D}. \end{aligned} \tag{9.30}$$

Combining the estimates (9.29) and (9.30) we get the inequality (9.8).

From Lemma 9.1 it follows that for sufficiently small T and r_0 the nonlinear operator S satisfies the conditions of the fixed point theorem for a contraction operator. Hence, we have proved the following theorem:

Theorem 9.1. *Assume that* $s = 1+\alpha$, (1.5) *and* (1.10) *hold for* $k = 7$, $\kappa(\Gamma) \in E^{5+\alpha}_{5+\alpha}(\Gamma)$, *and* (2.5) *holds. Then there exists a unique solution of the system* (9.1)–(9.5) *for small interval* $0 < t < T$, *such that*

$$w \in E^{2+\alpha,\alpha/3,\alpha}_{2+\alpha}(\Omega_T), \quad \sigma \in N^{4+\alpha}_{1+\alpha}(\Gamma_T).$$

Theorem 9.1 means that the original Hele–Shaw problem (1.1), (1.4), (1.7)–(1.9) has a unique smooth solution for small time.

Remark 9.1. Note that under our assumptions the initial corner points do not change in time. This is in agreement with the results from [9] for angular domains without surface tension.

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